ON THE USE OF BAYES' THEOREM IN ESTIMATING FALSE ALARM RATES

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1. INTRODUCTION

Epstein [2] has suggested that Bayes' theorem may have applications in forecast verification. I would like to suggest one such use, namely in estimating false alarm rates for forecasts of rare but hazardous events such as severe winds, earthquakes, tsunamis, major solar flares, etc. In particular the analysis based on Bayes' theorem provides some insight into the problem of why such forecasts usually show a high false alarm rate.

One of the difficulties in verifying rare event forecasts is that the forecasts are often issued only on an as-needed basis, and it is often difficult to determine whether the absence of a warning means a forecast of non-occurrence or simply that the system was temporarily inoperative. For the present purpose we will assume that there is a routine yes-no type forecast issued for a specified forecast interval, such as once per day, once per hour, etc. We will define the event E to mean that a forecast interval contains at least one instance of the event in question. We will assume that a set of antecedent conditions, A, had been determined that had some predictive association with E. We further assume that we have a reasonably good estimate of the probability of E, which we define as

$$P(E)=\lim_{N\to\infty}\frac{N_E}{N}$$

Where N_E is the number of occurrences of E and N is the number of forecast intervals.

We can define four forecast scores, using the nomenclature of Brier and Allen [1], which yield information on the degree of success of the type of forecasts being discussed here. The definitions are stated below and each is followed on the right by a clarifying expression in terms of the entries a, b, c, d of a model contingency table (table 1):

Prefigurance on Yes Forecasts

$$=\frac{\text{Number of Correct Yes Forecasts}}{\text{Number of Occurrences of } E}=\frac{a}{a+b}$$

Prefigurance on No Forecasts

$$=\frac{\text{Number of Correct No Forecasts}}{\text{Number of Occurrences of } E'}=\frac{d}{c+d}$$

where E′ means the non-occurrence of E.

TABLE 1.—*Model contingency table*

		Forecast: Yes	Forecast: No	Total
Observed	Yes	a	b	$a+b$
	No	c	d	$c+d$
	Total	$a+c$	$b+d$	N

Post agreement on Yes Forecasts

$$=\frac{\text{Number of Correct Yes Forecasts}}{\text{Number of Yes Forecasts}}=\frac{a}{a+c}$$

Post Agreement on No Forecasts

$$=\frac{\text{Number of Correct No Forecasts}}{\text{Number of No Forecasts}}=\frac{d}{b+d}$$

Although not mentioned by Brier and Allen, the term "False Alarm Rate" has been used frequently in discussing such forecasts. We define it here as:

$$\text{False Alarm Rate}=1-\text{Post Agreement on Yes}=1-\frac{a}{a+c}=\frac{c}{a+c}$$

If we let each of the above expressions on the right approach a limit as the denominator approaches infinity, we can now redefine the four scores as conditional probabilities as follows, where the symbol A means the event that the antecedent conditions were observed prior to the forecast period and A' means the non-occurrence of A:

$$\text{Prefigurance on Yes}=\frac{P(EA)}{P(E)}=P(A|E)$$

$$\text{Prefigurance on No}=\frac{P(E'A')}{P(E')}=P(A'|E')$$

$$\text{Post Agreement on Yes}=\frac{P(EA)}{P(A)}=P(E|A)$$

$$\text{Post Agreement on No}=\frac{P(E'A')}{P(A')}=P(E'|A')$$

The symbols used in the above definitions are essentially

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those used by Feller [3] and their meaning can be illustrated as follows:

$P(EA)$ is the probability of the joint occurrence of E and A. To a mathematician this joint occurrence means the intersection of E and A; to a forecaster this means that when the antecedent conditions were met, the event E did in fact occur subsequently.

$P(E|A)$ means the conditional probability of E, given that A occurred.

2. DERIVATION OF BAYES' THEOREM

We can now define the false alarm rate as follows:

$$P(E'|A)=\frac{P(E'A)}{P(A)}$$

But since A is associated with either the occurrence or non-occurrence of E,

$$P(E'|A)=\frac{P(E'A)}{P(E'A)+P(EA)}$$

Now, eliminating intersections by means of the definition of conditional probability, we have

$$P(E'|A)=\frac{P(A|E')P(E')}{P(A|E')P(E')+P(A|E)P(E)}$$

which is a two-event form of Bayes' theorem. Finally, to limit our formula only to expressions which we have defined in words, we write:

$$P(E'|A)=\frac{[1-P(A'|E')]P(E')}{[1-P(A'|E')]P(E')+P(A|E)P(E)}$$

3. COMPUTATION OF FALSE ALARM RATES

It turns out in the computation of false alarm rates for rare events that the value of $P(A|E)$ chosen makes little difference as long as it is reasonably high. In table 2 then we have assumed that $P(A|E)=0.95$, although we would have gotten almost identical values by letting it equal 1. Choosing lower values of $P(A|E)$ would, of course, serve to strengthen our main result, which is that high false alarm rates are inevitable in forecasting rare events.

TABLE 2.—*False alarm rates for various values of P(E) and Prefigurance on No, assuming Prefigurance on Yes=0.95*

$P(E)$	$P(A'\|E')$ = Prefigurance on No							
	0.90	0.95	0.96	0.97	0.98	0.99	0.995	0.999
0.005	0.95	0.91	0.89	0.86	0.81	0.67	0.51	0.17
.01	.91	.84	.81	.76	.68	.51	.34	.09
.02	.84	.72	.67	.61	.51	.34	.20	.05
.04	.72	.56	.50	.43	.34	.20	.11	.02

TABLE 3.—*Contingency table of hypothetical Yes-No forecasts*

Observations	Predictions: Yes	No	Totals
Yes	19	1	20
No	199	3781	3980
Totals	218	3782	4000

Table 2 shows that if we consider a rare event to be one that occurs in 1 out of 25 or more forecast intervals and if we agree that forecast studies do not generally yield values of $P(A'|E')$ much greater than 0.95, we are forced to accept false alarm rates of greater than 50 percent. This does not mean that such a forecast performance is lacking in skill or in usefulness. In situations where the penalty for operating under hazardous conditions is great, the user should not, if properly educated, be disappointed by a false alarm rate of even as high as 90 percent.

Table 3 shows a hypothetical set of verification statistics for assumed values of $P(E)=0.005$, $P(A|E)=0.95$, $P(A'|E')=0.95$. We see from this table that the false alarm rate is 91 percent. Yet the prefigurance on Yes, prefigurance on No, and percentage of "hits" are all equal to 0.95. A user receiving this type of service should be reminded that he is getting excellent forecast service. He was allowed to operate with virtually 100 percent safety in 94.5 percent of the cases.

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REFERENCES

1. G. W. Brier, and R. A. Allen, "Verification of Weather Forecasts," *Compendium of Meteorology*, (T. Malone, ed.), American Meteorological Society, 1951, pp. 841–848.
2. E. S. Epstein, "A Bayesian Approach to Decision Making in Applied Meteorology," *Journal of Applied Meteorology*, vol. 1, No. 2, June, 1962, pp. 169–177.
3. W. Feller, *An Introduction to Probability Theory and Its Application*, John Wiley and Sons, New York, 1957.

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